Materials Today • Volume xxx, Number xx • xxxx 2021 RESEARCH

On the expanded Maxwell's equations for moving charged media system – General theory, mathematical solutions and applications in TENG

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The conventional Maxwell's equations are for media whose boundaries and volumes are fixed. But for cases that involve moving media and time-dependent configuration, the equations have to be expanded. Here, starting from the integral form of the Maxwell's equations for general cases, we first derived the expanded Maxwell's equations in differential form by assuming that the medium is moving as a rigid translation object. Secondly, the expanded Maxwell's equations are further developed with including the polarization density term $\boldsymbol{P}_s$ in displacement vector owing to electrostatic charges on medium surfaces as produced by effect such as triboelectrification, based on which the first principle theory for the triboelectric nanogenerators (TENGs) is developed. The expanded equations are the most comprehensive governing equations including both electromagnetic interaction and power generation as well as their coupling. Thirdly, general approaches are presented for solving the expanded Maxwell's equations using vector and scalar potentials as well as perturbation theory, so that the scheme for numerical calculations is set. Finally, we investigated the conservation of energy as governed by the expanded Maxwell's equations, and derived the general approach for calculating the displacement current $\frac{\partial}{\partial t}\boldsymbol{P}_s$ for the output power of TENGs. The current theory is general and it may impact the electromagnetic wave generation and interaction (reflection) with moving train/car, flight jets, missiles, comet, and even galaxy stars if observed from earth.

Keywords: Expanded Maxwell's equations; Displacement current; Moving charged media; Triboelectric nanogenerator

Introduction

Maxwell's equations are probably the top #1 equations for the field of physics, which have huge importance in fundamental science and practical technologies [1]. Starting from experimentally observed physics laws, such as Faraday's electromagnetic induction law, Ampere's law, Maxwell's equations unified the electricity and magnetism, which later inspires the advocating of unifying the four forces in nature. The electromagnetic wave theory and the wireless communication technologies established based on Maxwell's equations are the foundation of modern telecommunication. Just like many others, we learnt Maxwell's equations mainly through standard text books that does not specifically devote much text for introducing the background and assumptions made for deriving the Maxwell's equations. In the famous Jackson's book on "*Classical Electrodynamics*", only a couple of pages were devoted to the first introduction of displacement current, which is then fully integrated in the Maxwell's equations without in-depth exploration [2]. This is probably the reason that the main objective of developing the Maxwell's equations was for the purpose of studying the behavior of electromagnetic wave and its interaction with matter.

Recently, the concept of displacement current has been applied for quantifying the power output of piezoelectric and

TENG, Triboelectric nanogenerator.

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1369-7021/ https://doi.org/10.1016/j.mattod.2021.10.027

triboelectric nanogenerators (TENGs) [3–5], which is emerging as a new technology for fully utilizing high entropy energy [6], which is the energy that is widely distributed in our living environment with low quality, low amplitude and even low frequency. TENG was first invented in 2012, and it has four basic working modes: the contact-separation mode, lateral sliding mode, single-electrode mode, and free-standing mode (see Fig. 1) [7,8]. TENG has a broad application as micro-nano power source, self-powered sensors, blue energy and high voltage sources, covering area from medical science, wearable electronics, flexible electronics, security, human–machine interfaces and even environmental science [7,9–15]. Let's take the contact-separation mode TENG shown in Fig. 1a as an example. Under the mechanical pull-press force acting in vertical direction, the two dielectric layers are periodically contacted and separated. The two surfaces have opposite electrostatic charges owing to contact electrification effect. A change in spatial distribution of the media, surface electrostatic charge density, as well as the distance between the two electrodes, results in a variation of electric field in space, which is a form of displacement current that generates an output conduction current across the load connected between the two electrodes. As a general case, the media boundaries here do vary with time, and we need to derive the Maxwell's equations for moving charged media.

Maxwell first added into the induction equation the term that described the induction due to the motion of the medium [16]. Hertz systematically extended Maxwell's theory for moving media [17] but his equations were valid only for conductors and needed to be expanded on the cases of dielectrics and empty space. Minkowski derived electrodynamic equations for moving media using the principle of relativity [18]. However, the development of electrodynamics for moving media was almost interrupted due to the appearance of theory of relativity. In recent years, the interest on the study of electrodynamics of moving media has been revived [19,20]. There are a number of studies about the Maxwell's equations for moving media/bodies with a focus on the scattering, reflection and transmission of electromagnetic waves from moving media [21–23]. But most of these studies are mainly focused on the establishment of the first principle equations without much progress proposed for analytical solutions of the equations and their practical applications.

In this paper, starting from the integral form of the Maxwell's equations, we first derived the standard differential form of the Maxwell's equations by assuming that the media volumes and surfaces/interfaces are fixed. Secondly, we derived the expanded Maxwell's equations by assuming that the medium is moving as a rigid translation. It is important to point out the differences between our expansion of Maxwell's equations presented here

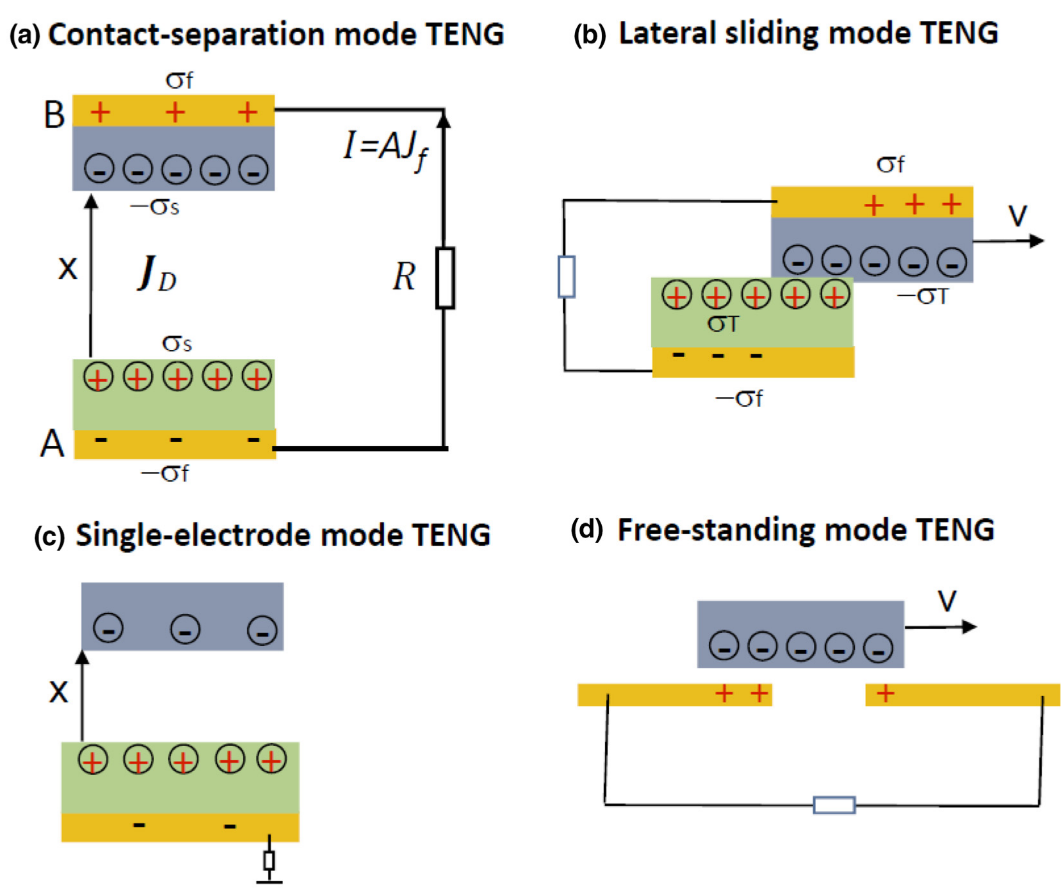

FIGURE 1

Schematics showing the four modes of tribielectric nanogenerators based on coupling effect of triboelectrification and electrostatic induction, for effectively harvesting high entropy energy in our living environment.

Please cite this article in press as: Z.L. Wang, Materials Today, (2021), https://doi.org/10.1016/j.mattod.2021.10.027

from special relativity. Special relativity is the theory of how different observers, moving at constant velocity with respect to one another, report their experience of the same physical event. Our theory is about the observation of the electromagnetic behavior of the system in a stationary coordination frame when some of the charged media are moving at different speeds, and there may have interaction and/or charges/current exchange between the media that are at rest and in moving (see the case in Fig. 1 for example). Thirdly, we expanded the equations for including the cases in which the media surfaces have electrostatic charges that were produced by triboelectrification or piezoelectric effect. An addition of a polarization term $\boldsymbol{P}_s$ in the displacement vector that arises from the surface electrostatic charges is the most logic approach for dealing with the first principle theory of the TENGs. Such equations are the most comprehensive governing equations including both electromagnetic interaction and power generation. Fourthly, we investigated the conservation of energy as governed by the expanded Maxwell's equations, and derived the general approach for calculating the displacement current $\frac{\partial}{\partial t}\boldsymbol{P}_s$. Finally, general strategies for solving the expanded Maxwell's equations are proposed.

Medium polarization and the law of charge conservation

We first introduce the basics of the electrodynamics. The existence of free charges in space would produce an electric field $\boldsymbol{E}$, the presence of which causes the bound charges in the material (atomic nuclei and their electrons) to slightly separate, inducing a local electric dipole moment, which is called polarization. If all of the dipoles in the medium add up, a macroscopic polarization would be observed in space, which counts for the screening of the medium to the free charges, normally called dielectric screening effect. As a result the electric displacement field $\boldsymbol{D}'$ is defined as

$$\boldsymbol{D}' = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} \tag{1}$$

where ε_0 is the vacuum permittivity, and $\boldsymbol{P}$ is the density of the permanent and induced electric dipole moments in the media as a result of the applied electric field $\boldsymbol{E}$, called the polarization density. If the space charge density of free charges is ρ_f, and the $\boldsymbol{P}$ is polarization density caused by the induced bond charge density ρ_b,

$$\rho_b = -\nabla \cdot \boldsymbol{P}. \tag{2}$$

The total charge density in space would be:

$$\rho = \rho_f + \rho_b = -\varepsilon_0 \nabla \cdot \boldsymbol{E} \tag{3}$$

Therefore, $\nabla \cdot \boldsymbol{D}' = \rho_f$, which is the Gauss law for electric displacement field. Since ρ_f makes the volume non-neutral, the medium is responded with a polarization charge ρ_b, which is the density of all those charges that are part of a dipole, each of which is neutral.

For simplicity, we mainly assume that the dielectric media we are interested in here are isotropic materials, so that its local polarization density is

$$\boldsymbol{P} = \varepsilon_0 \chi \boldsymbol{E}, \tag{4}$$

where χ is the electric susceptibility. The bond charge density is

$$\rho_b = -\nabla \cdot \boldsymbol{P} = -\chi(\rho_f + \rho_b) = -\rho_f\, \chi/(\chi + 1). \tag{5}$$

Correspondingly, the surface bound electrostatic charge density is $\sigma_b = \boldsymbol{n} \cdot \boldsymbol{P}$, where n is the surface normal direction pointing outside from the medium. The above discussion means that the introduction of $\boldsymbol{P}$ is to account for the contribution of the induced bond charges to the local electric field.

An important law for electromagnetism is the conservation of charges. The local *free charge density* and the local *free electric current density* $\boldsymbol{J}_f$ must satisfy:

$$\nabla \cdot \boldsymbol{J}_f + \frac{\partial}{\partial t}\rho_f = 0. \tag{6}$$

where the first term is the divergence of the free current density that represents the current going into and coming out of the surface, and the second term is the changing rate of the free charge density.

The displacement current

The most conventional current that we are familiar with is the conduction current that is the result of electron flow in conducting medium as driven by an electric field. Besides, there is another type of current called displacement current. We now review the process for introducing the displacement current by Maxwell in 1861. From the Ampère's law:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_f \tag{7}$$

which is a relationship between the magnetic field generated by a flowing conduction current. Mathematically, one must have

$$\nabla \cdot (\nabla \times \boldsymbol{H}) = \nabla \cdot \boldsymbol{J}_f = 0 \tag{8}$$

which is apparently incorrect, because $\nabla \cdot \boldsymbol{J}_f = -\frac{\partial}{\partial t}\rho_f = -\frac{\partial}{\partial t}\nabla \cdot \boldsymbol{D}' = -\nabla \cdot \frac{\partial}{\partial t}\boldsymbol{D}' \neq 0$. Therefore, to satisfy the law of conservation of charges, one term must be added in the current, so that the Ampere's law is modified as [24]:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial \boldsymbol{D}'}{\partial t} \tag{9}$$

where the term $\frac{\partial \boldsymbol{D}'}{\partial t} = \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \frac{\partial \boldsymbol{P}}{\partial t} = \varepsilon \frac{\partial \boldsymbol{E}}{\partial t}$ is called the displacement current, which is fundamental for not only unifying the electricity and magnetism, but also set the foundation for the electromagnetic wave and its transmission. Eq. (9) is thus referred as the Ampere-Maxwell's law. Therefore, according to Maxwell, the displacement current is not an electric current of moving charges, but a time-varying electric field $\left(\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right)$, plus a contribution from the slight motion of charges bounded in atoms $\left(\frac{\partial \boldsymbol{P}}{\partial t}\right)$. This means that the displacement current first introduce by Maxwell only has one type: the time variation term $\left(\varepsilon \frac{\partial \boldsymbol{E}}{\partial t}\right)$.

From integral form to differential form of the Maxwell equations for time-independent medium configuration

The integral form of the Maxwell's equation is general that is a direct result of the physics laws and is directly related to the experimentally observed physics phenomena, such as electromagnetic induction. We start from the integral form of the Maxwell's equations:

$$\oint\!\!\oint_S \boldsymbol{D}' \cdot d\boldsymbol{s} = \iiint_V \rho_f\, d\boldsymbol{r} \text{ (Gauss law for electricity)} \tag{10a}$$

$$\oint\!\!\oint_S \boldsymbol{B} \cdot d\boldsymbol{s} = 0 \text{ (Gauss law for magnetism)} \tag{10b}$$

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{L} = -\frac{d}{dt}\iint_C \boldsymbol{B} \cdot d\boldsymbol{s} \text{ (Faraday's Law of electromagnetic induction)} \tag{10c}$$

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$$\oint_C \boldsymbol{H}\cdot d\boldsymbol{L} = \iint_C \boldsymbol{J}_f \cdot d\boldsymbol{s} + \frac{d}{dt}\iint_C \boldsymbol{D}' \cdot d\boldsymbol{s} \text{ (Ampere - Maxwell law)} \tag{10d}$$

where the surface integrals for $\boldsymbol{B}$ and $\boldsymbol{D}'$ are for a surface that is defined by a closed loop c, and they are the magnetic flux and displacement field flux, respectively. The law of the conservation of charges is:

$$\oiint_S \boldsymbol{J}_f \cdot \mathrm{d}\boldsymbol{s} + \frac{d}{dt}\iiint_V \rho_f\, \mathrm{d}\boldsymbol{r} = 0 \tag{10e}$$

The integral form of the Maxwell's equations are more general, but the most commonly used Maxwell's equations are in differential form. Now we make an important assumption: the volume and shape/boundaries of the dielectric media in space are independent of time. Under such an assumption for fixed boundaries, the time differentiation can be directly applied to the corresponding function inside the integral. By applying the basic divergence theorem, Stokes's theorem,

$$\oint_C \boldsymbol{a}\cdot d\boldsymbol{L} = \iint_C \nabla\times\boldsymbol{a}\cdot d\boldsymbol{s} \tag{11a}$$

$$\oiint_s \boldsymbol{a}\cdot d\boldsymbol{s} = \iiint_V \nabla\cdot\boldsymbol{a}\, d\boldsymbol{r} \tag{11b}$$

from Eqs. (10a–d), we have

$$\iiint_V \nabla\cdot\boldsymbol{D}'\, d\boldsymbol{r} = \iiint_V \nabla\cdot\rho_f\, d\boldsymbol{r} \tag{12a}$$

$$\oiint_s \boldsymbol{B}\cdot d\boldsymbol{s} = 0 \tag{12b}$$

$$\iint_C (\nabla\times\ \boldsymbol{E})\cdot \mathrm{d}\boldsymbol{s} = -\iint_C \frac{\partial}{\partial t}\boldsymbol{B}\cdot d\boldsymbol{s} \tag{12c}$$

$$\iint_C (\nabla\times\ \boldsymbol{H})\cdot d\boldsymbol{s} = \iint_C \boldsymbol{J}_f\cdot d\boldsymbol{s} + \iint_C \frac{\partial}{\partial t}\boldsymbol{D}'\cdot d\boldsymbol{s} \tag{12d}$$

and consider the shape and boundaries referred above are for arbitrary objects, so that the functions inside the integral must satisfy:

$$\nabla\cdot\boldsymbol{D}' = \rho_f \tag{13a}$$

$$\nabla\cdot\boldsymbol{B} = 0 \tag{13b}$$

$$\nabla\times\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B} \tag{13c}$$

$$\nabla\times\boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial}{\partial t}\boldsymbol{D}' \tag{13d}$$

This is the most familiar form of Maxwell's equations that we use for many applications. However, one must point out that *the differential form of Maxwell's equations applies only to the cases in which the volumes and boundaries of the dielectric media are time-independent*, which means that the boundaries and distribution configurations of the dielectrics are fixed. This is the case if one is only interested in the generation and transmission of electromagnetic waves for stationary media! One has to keep this in mind because it is rarely mentioned in text books.

From integral form to differential form of the Maxwell equations for time-dependent medium configuration

Alternatively, in a case that the volume and boundaries of the media vary with time, especially with the triggering of external forces $\boldsymbol{F}$ (Fig. 2). The mathematics for such cases are rather complex. For simplicity, here we consider a case in which the dielectric medium is assumed to be a group of rigid objects whose shapes and surfaces do not vary with time, but experiencing a rigidly translation following a trajectory of the centroid described by $\boldsymbol{r}_0(t)$ as for the dielectric medium. The translation velocity is $\boldsymbol{v} = d\boldsymbol{r}_0(t)/dt$. In such a case, the coordinates for the two reference frames affixed to the medium and at the original frame are: $\boldsymbol{r} = \boldsymbol{r}_0(t) + \boldsymbol{r}_v$.

Our goal here is to find out the electromagnetic behavior of a system in a stationary coordination frame in which the media are moving with respect to each other, and different media could move at different speeds. In Fig. 2, in the observer's coordination frame, medium A remains stationary, medium B is moving, and a medium C, if exists, could move at a different speed. Such a case is different from the situation for special relativity, which is about how different observers, moving at constant velocity with respect to one another, report their experience of the same physical event. For simplicity, the relativistic effect is not considered in following derivation by assuming $v \ll c$ (speed of light), which is an excellent approximation for almost all of the moving media. All of our discussions hereafter are for low moving speed object $v \ll c$, so that the special relativity effect is not included.

We first introduce two mathematical identities for general functions $g(\boldsymbol{r},t)$ and $\boldsymbol{G}(\boldsymbol{r},t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V g\,\mathrm{d}\boldsymbol{r} = \iiint_V \left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right) g\,\mathrm{d}\boldsymbol{r} = \iiint_V \frac{\partial}{\partial t} g\,\mathrm{d}\boldsymbol{r} + \oiint_s g\ \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{s}. \tag{14a}$$

$$\begin{aligned}\frac{d}{dt}\iint_C \boldsymbol{G}\cdot d\boldsymbol{s} &= \iint_C \left[\frac{\partial}{\partial t}\boldsymbol{G} + (v\cdot\nabla)\boldsymbol{G}\right]\cdot d\boldsymbol{s} \\ &= \iint_C \left[\frac{\partial}{\partial t}\boldsymbol{G} + (\boldsymbol{G}\cdot\nabla)v + (\nabla\cdot\boldsymbol{G})v - (\nabla\cdot v)\boldsymbol{G}\right]\cdot d\boldsymbol{s} \\ &\quad - \oint (v\times\boldsymbol{G})d\boldsymbol{L}\end{aligned} \tag{14b}$$

The corresponding Maxwell's equations are given by:

$$\nabla\cdot\boldsymbol{D}' = \rho_f \tag{15a}$$

$$\nabla\cdot\boldsymbol{B} = 0 \tag{15b}$$

$$\nabla\times\boldsymbol{E} = -\frac{D}{Dt}\boldsymbol{B} \tag{15c}$$

$$\nabla\times\boldsymbol{H} = \boldsymbol{J}_{\mathrm{f}} + \frac{D}{Dt}\boldsymbol{D}' \tag{15d}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\boldsymbol{v}\cdot\nabla) \tag{15e}$$

The law of charge conservation is:

$$\oiint_S \boldsymbol{J}_f\cdot\mathrm{d}\boldsymbol{s} + \iiint_V \frac{\partial}{\partial t}\rho_f\,\mathrm{d}\boldsymbol{r} + \oiint_S \rho_f\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{s} = 0 \tag{16a}$$

$$\nabla\cdot\boldsymbol{J}_f + \frac{D}{Dt}\rho_f = 0,\ \text{or}\ \nabla\cdot(\boldsymbol{J}_f + \rho_f\boldsymbol{v}) + \frac{\partial}{\partial t}\rho_f = 0. \tag{16b}$$

where $\rho_f\boldsymbol{v}$ is the current produced by the free charges as the medium being translated at a velocity $\boldsymbol{v}$. The solution of Eqs. (15a–15d) using various approaches are given in the sections on Solutions of the vector

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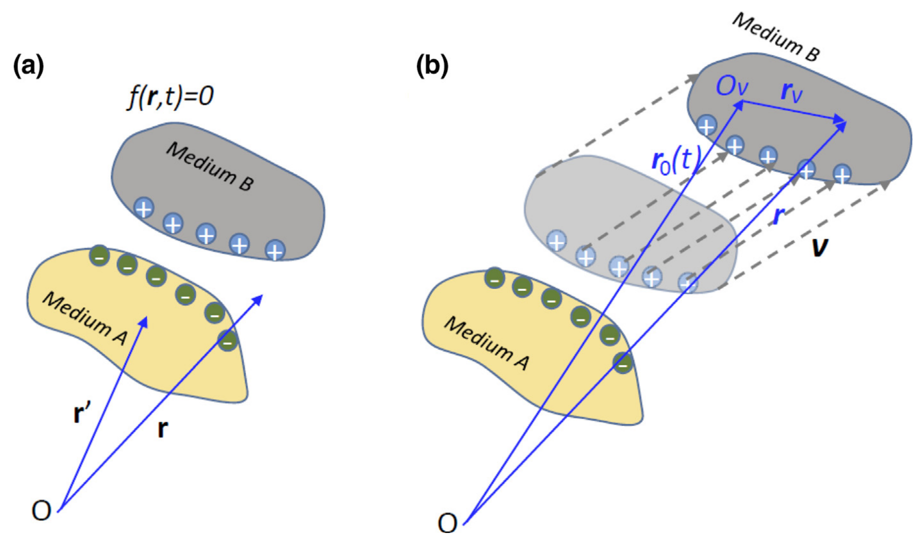

FIGURE 2

Schematic diagram showing the translation movement of dielectric media in space with speed $\boldsymbol{v}$ under the driving of an external force $\boldsymbol{F}$. The shape of the medium surface is defined by $f(\mathbf{r},t) = 0$. The electrostatic charges due to triboelectrification effect, for example, are schematically shown. $\mathbf{r}_0$ is the origin of the reference frame O_v affixed to medium B that is translating at a speed of $\boldsymbol{v}$, and $\mathbf{r}_v$ is the coordination system in this reference frame.

and scalar potentials Solution of the expanded Maxwell's equations in frequency space.

Now for a case in which the moving velocity of the rigid medium is a constant that is independent of (x, y, z), using mathematical identity:

$$\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a}(\nabla \cdot \boldsymbol{b}) - \boldsymbol{b}(\nabla \cdot \boldsymbol{a}) + (\boldsymbol{b} \cdot \nabla)\boldsymbol{a} - (\boldsymbol{a} \cdot \nabla)\boldsymbol{b} \tag{17}$$

Eqs. (15a–d) become

$$\nabla \cdot \boldsymbol{D}' = \rho_f \tag{18a}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{18b}$$

$$\nabla \times (\boldsymbol{E} - \boldsymbol{v} \times \boldsymbol{B}) = -\frac{\partial}{\partial t}\boldsymbol{B} \tag{18c}$$

$$\nabla \times (\boldsymbol{H} + \boldsymbol{v} \times \boldsymbol{D}') = \boldsymbol{J}_f + \rho_f \boldsymbol{v} + \frac{\partial}{\partial t}\boldsymbol{D}' \tag{18d}$$

where $\rho_f \boldsymbol{v}$ is the current due the medium translation movement. Eqs. (18a–d) satisfy the charge conservation law Eq. (16b). It is noticed that the translation movement of the media results in a small correction to the local electric and magnetic fields due to electro-magnetic coupling as a result of medium movement [25]. General equations for non-uniform moving media case has been given by Kaufman [26].

Accordingly, the boundary conditions for Eqs. (18a–d) are:

$$\left[\boldsymbol{D}'_2 - \boldsymbol{D}'_1\right] \cdot \boldsymbol{n} = \sigma_f \tag{19a}$$

$$[\boldsymbol{B}_2 - \boldsymbol{B}_1] \cdot \boldsymbol{n} = 0 \tag{19b}$$

$$\boldsymbol{n} \times [\boldsymbol{E}_2 - \boldsymbol{E}_1 - \boldsymbol{v} \times (\boldsymbol{B}_2 - \boldsymbol{B}_1)] = 0 \tag{19c}$$

$$\boldsymbol{n} \times \left[\boldsymbol{H}_2 - \boldsymbol{H}_1 + \boldsymbol{v} \times (\boldsymbol{D}'_2 - \boldsymbol{D}'_1) = \boldsymbol{K}_s + \sigma_f\ \boldsymbol{v}_s\right] \tag{19d}$$

where $\boldsymbol{K}_s$ is the surface current density, σ_f is the surface free charge density, and $\boldsymbol{v}_s$ is the moving velocity of the media in parallel to the boundary boundary.

Polarization introduced by moving charged boundary/media

In the cases for electromagnetism, the medium boundary and media shape are usually assumed independent of time. Traditionally, the Maxwell's equations are exclusively used for describing the interaction of electromagnetic wave with media and the generation, transmission and receiving of electromagnetic wave, in which the shape of the antenna rarely changes, so that the boundaries associated mathematics is time-independent. However, for energy conversion, external mechanical triggering usually causes the dielectric media to change in shape or distribution, so that the configuration and boundary conditions are time-dependent. The electrostatic charges on surfaces can be due to triboelectric or piezoelectric effect as for the case of nanogenerators. Now let's consider a case, in which the medium is "precharged" with electrostatic charges, thus, a variation in medium shape and/or moving medium object results in not only a local time-dependent charge density ρs, but also a local "virtual" electric current density due to the 'passing-by' of the electrostatic charges on the surface of the object once it moves (Fig. 1). To account both terms, the displacement vector has to be modified by adding an additional term $\boldsymbol{P}_s$, representing the polarization owing to the pre-existing electrostatic charges on the media, so that the displacement vector is modified as [4]

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} + \boldsymbol{P}_s = \varepsilon_0(1 + \chi)\boldsymbol{E} + \boldsymbol{P}_s \tag{20}$$

Here, the first term ε_0E is due to the field created by the free charges, called external electric field; the polarization vector $\boldsymbol{P}$ is the medium polarization caused by the existence of the external electric field $\boldsymbol{E}$; and the added term $\boldsymbol{P}_s$ is mainly due to the existence of the surface electrostatic charges and the time variation in boundary shapes. The corresponding space charge density is

$$\rho_s = -\nabla \cdot \boldsymbol{P}_s; \tag{21a}$$

the surface electrostatic charge density is $\sigma_s = \boldsymbol{n} \cdot \boldsymbol{P}_s$; and the current density contributed by the bond electrostatic charges is

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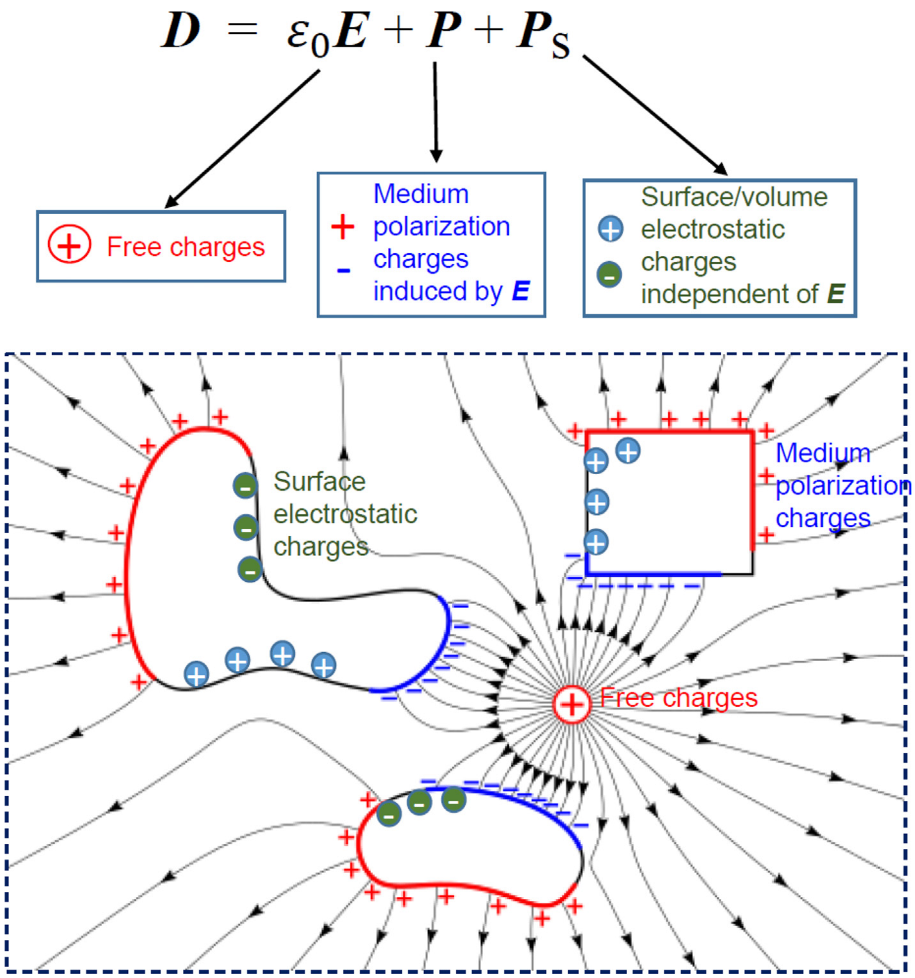

FIGURE 3

Schematic showing the three terms in the newly defined displacement vector $\boldsymbol{D}$, and their represented space charges in the diagram. The charge density corresponding to $\boldsymbol{P}_s$ is that from surface contact electrification effect in TENG.

$$\boldsymbol{J}_s = \frac{\partial}{\partial t}\boldsymbol{P}_s. \tag{21b}$$

The physical meaning of the terms in Eq. (20) can be explained using Fig. 3 as follows. The charges that create the first term $\varepsilon_0\boldsymbol{E}$ is called free charges, which is the field for exciting the media. The polarizations produced by the electric field $\boldsymbol{E}$ results in a local polarization $\boldsymbol{P}$, which is responsible for the screening effect of the medium to the external electric field $\boldsymbol{E}$. If the surface of the medium has electrostatic charges that are produced by effects such as piezoelectric effect and/or triboelectric effect, an additional term $\boldsymbol{P}_s$ is added in displacement vector $\boldsymbol{D}$. The charges that contribute to the P_s term are neither free charges, not polarization induced charges, instead they are intrinsic surface bound electrostatic charges as introduced by external mechanical triggering to the media. This term is necessary for developing the theory of mechanical to electric energy conversion.

Expanded Maxwell's equations for moving charged media

General approach

If we consider that the surfaces of the dielectric medium have electrostatic charges owing to effects such as piezoelectricity and triboelectricity, according to Eqs. (20a–b). The presence of the surface electrostatic charges will make a substitution of $\boldsymbol{D}'$ by $\boldsymbol{D} = \boldsymbol{D}' + \boldsymbol{P}_s$ in Eqs. (18a–d):

$$\nabla \cdot \boldsymbol{D}' = \rho_f - \nabla \cdot \boldsymbol{P}_s \tag{22a}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{22b}$$

$$\nabla \times (\boldsymbol{E} - \boldsymbol{v} \times \boldsymbol{B}) = -\frac{\partial}{\partial t}\boldsymbol{B} \tag{22c}$$

$$\nabla \times \left[\boldsymbol{H} + \boldsymbol{v} \times (\boldsymbol{D}' + \boldsymbol{P}_s)\right] = \boldsymbol{J}_f + \rho_f \boldsymbol{v} + \frac{\partial}{\partial t}\boldsymbol{D}' + \frac{\partial}{\partial t}\boldsymbol{P}_s \tag{22d}$$

Eqs. (22a–d) are not only self-consistent, but also satisfy the charge conservation law as defined in Eq. (16b).

Eqs. (22a–d) can be easily understood in comparison to Eqs. (18a–d) equivalently by a substitution:

$$\rho_f \rightarrow \rho_f - \nabla \cdot \boldsymbol{P}_s \quad \text{as the total ``free'' charge density;} \tag{23a}$$

$$\boldsymbol{J}_f + \rho_f \boldsymbol{v} \rightarrow \boldsymbol{J}_f + \rho_f \boldsymbol{v} + \frac{\partial}{\partial t}\boldsymbol{P}_s \quad \text{as the total ``free'' current density.} \tag{23b}$$

Such substitution not only includes the contributions from all terms, but also warranty the satisfaction of the conservation of charges. Eqs.

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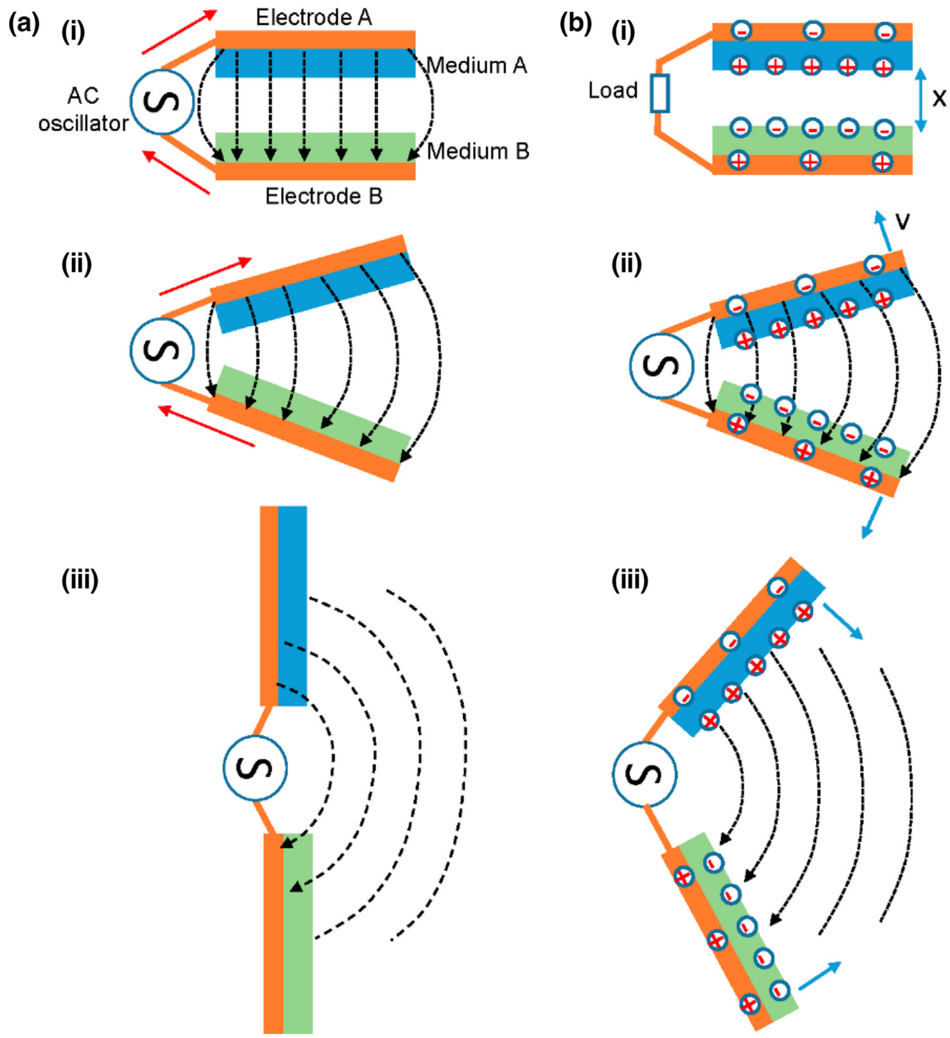

FIGURE 4

(a) Starting from a simple capacitor model made of two parallel plates, an AC oscillating current source will produce an alternating electric field across the two electrodes (a-i). The leaked field will increase with the opening of the plates (a-ii), and eventually radiates to a long distance if the plates are fully opened (a-iii). (b) By combining the capacitor model in (a) with the integration of two dielectric layers inside, forming a contact-separation mode TENG, the coupling between electromagnetic wave radiation and the triboelectric nanogenerator as driven by cycled mechanical triggering is step by step demonstrated (b-i to b-iii). The fully electrodynamics for this case is covered by the expanded Maxwell's equations (Eqs. (22a–d)).

(22a–d) are the fully expanded Maxwell equation for a moving charged media whose surface has electrostatic charges.

It is relatively easy to understand the foundation of the Maxwell's equations for electromagnetic wave. Now let's use Fig. 4 to illustrate the coupling between electromagnetic wave and the energy conversion process as governed by Eqs. (22a–d). Fig. 4 shows the basic process of creating electromagnetic radiation by an oscillating AC source. If one applies an AC oscillating current to the two parallel metal plates of a capacitor, an electric field would be built up inside the capacitor, and a small field leakage is possible at the edge of the plates (Fig. 4a-i). Once the two metal plates are opened to form a fan shape (Fig. 4a-ii), the leaked electric field is more pronounced at the open end. The time-dependent electric field across the metal plates would produce a time-dependent magnetic field according to the Ampere-Maxwell's law. Once the two plates are fully opened as shown in Fig. 4a-iii, the AC generated electric field would propagate to a large distance. This is the process of electromagnetic radiation. The electrodynamics of this process is described by Eqs. (13a–d) if there is no medium movement and by Eqs. (18a–d) if there is medium movement.

Now let's use the contact-separation mode TENG to simulate the configuration of a capacitor for generating electromagnetic wave. The TENG can be viewed as a capacitor but with two different dielectric films attached to the inner side of the two electrodes. Besides the AC produced electric field across the two metal plates, the field produced by the dielectric media owing to the presence of triboelectric charges on the medium surface should be considered (Fig. 4b-i). Let's assume that the distance between the two dielectric surfaces being varied as driven by an externally applied periodic force, the leaked electric field has the contributions from both the electrodes and the triboelectric layers. If the "clamping" frequency of the two electrode layers is increased, the radiated electromagnetic waves are contributed by both the excitation current source Jf and the time-dependent variation of the electrostatic charges in space (Fig. 4-

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b-ii). This is a coupling result between the AC generated electromagnetic waves and the mechanical clamping created electromagnetic radiation ($\rho_f \boldsymbol{v} + \frac{\partial}{\partial t}\boldsymbol{P}_s$). Such coupling is possible if the mechanical operation frequency is in the range of MEMS or NEMS. All of these contributions are comprehensively included in Eqs. (22a–e).

A nanogenerator is made of dielectric media that produce the strain induced electrostatic charges on surfaces, the electrodes that have free charge distribution ρ_f, and interconnect conductive wire across the external load that carries the free-flowing current ($\boldsymbol{J}_f$) (see Fig. 1a). Once a mechanical agitation is acting on the media, the distribution and/or configuration of the electrostatic charges and media shapes vary with time, thus, an additional current density term $\frac{\partial \boldsymbol{P}_s}{\partial t}$ has to be introduced in the total current in order to account for such medium polarization.

From Eqs. (22a–e), the conduction current $\boldsymbol{J}_f$ is responsible for the AC current from the oscillator that generates an alternating magnetic field $\boldsymbol{H}$ (Eq. (22d)); the alternating $\boldsymbol{H}$ results in an alternating electric field $\boldsymbol{E}$ owing to the electromagnetic induction (Eq. (22c)) and the presence of the displacement current $\varepsilon\partial\boldsymbol{E}/\partial t$. Therefore, an electromagnetic wave ($\boldsymbol{E}$, $\boldsymbol{H}$) is generated in space. The total displacement current is:

$$\boldsymbol{J}_{\mathrm{D}} = \frac{\partial \boldsymbol{D}'}{\partial t} + \frac{\partial \boldsymbol{P}_s}{\partial t} = \varepsilon\frac{\partial \boldsymbol{E}}{\partial t} + \frac{\partial \boldsymbol{P}_s}{\partial t} \tag{24}$$

where $\frac{\partial \boldsymbol{D}'}{\partial t}$ represents the displacement current due to time variation of the electric field, and the term $\frac{\partial \boldsymbol{P}_s}{\partial t}$ is the current due to the movement of the changed media as driven by an external mechanical agitation/-force, which is referred as the Wang term. The total current in space is:

$$\begin{aligned}\boldsymbol{J}_{\mathrm{T}} &= \boldsymbol{J}_f + \rho_f\boldsymbol{v} + \boldsymbol{J}_{\mathrm{D}} = \boldsymbol{J}_f + \rho_f\boldsymbol{v} + \frac{\partial \boldsymbol{D}'}{\partial t} + \frac{\partial \boldsymbol{P}_s}{\partial t} \\ &= \boldsymbol{J}_f + \rho_f\boldsymbol{v} + \varepsilon\frac{\partial \boldsymbol{E}}{\partial t} + \frac{\partial \boldsymbol{P}_s}{\partial t}\end{aligned} \tag{25}$$

where the total displacement current J_D is responsible for the current observed in nanogenerator, and it is the core for converting mechanical energy into electric power. The term $\boldsymbol{J}_f$ is the conduction current received across a load that is connected to the electrodes of a nanogenerator (See Fig. 1a).

The variation of magnetic field term $\frac{\partial}{\partial t}\boldsymbol{B}$ in the Faraday's electromagnetic induction law is the fundamental for electromagnetic generator that converts mechanical energy into electricity. This has been the most important and widely used energy technology and power system. The displacement current $\varepsilon\frac{\partial \boldsymbol{E}}{\partial t} + \frac{\partial \boldsymbol{P}_s}{\partial t}$ in the Ampere–Maxwell law is the main driving force for the nanogenerator, which typically has a high output voltage and is especially effective for converting low quality and small amplitude mechanical energy into electric power. The relationship between the two types of power generators are summarized in Fig. 5.

Approximated results

If we ignore the corrections of the $\boldsymbol{v}$ terms made in the curl of the electromagnetic field $\boldsymbol{E}$ and magnetic field $\boldsymbol{H}$ since the media moving speed is rather small, Eqs. (22a–e) can be approximately written as:

$$\nabla \cdot \boldsymbol{D}' = \rho' \tag{26a}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{26b}$$

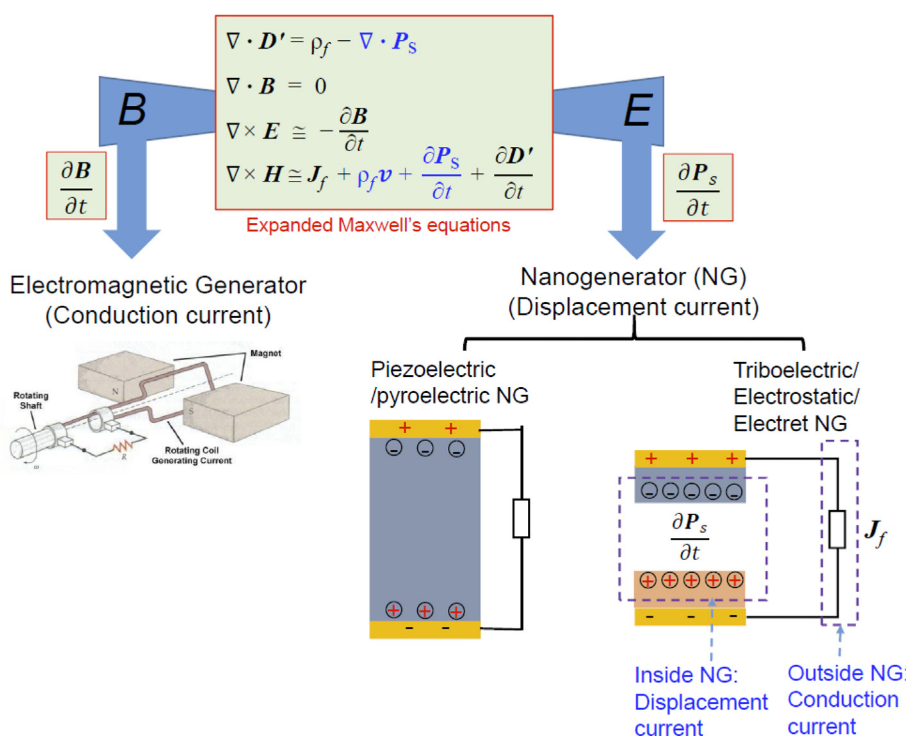

FIGURE 5

A comparison of traditional electromagnetic generator and triboelectric, piezoelectric, pyroelectric and electrostatic nanogenerators (NGs) regarding to the governing physics laws, types of currents, and their representing physical quantities in the expanded Maxwell's equations.

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$$\nabla \times \boldsymbol{E} \cong -\frac{\partial}{\partial t}\boldsymbol{B} \tag{26c}$$

$$\nabla \times \boldsymbol{H} \cong \boldsymbol{J}' + \frac{\partial}{\partial t}\boldsymbol{D}' \tag{26d}$$

where

$$\rho' = \rho_f - \nabla \cdot \boldsymbol{P}_s \tag{27a}$$

$$\boldsymbol{J}' = \boldsymbol{J}_f + \rho_f \boldsymbol{v} + \frac{\partial \boldsymbol{P}_s}{\partial t} \tag{27b}$$

which satisfy the law of conservation of charges:

$$\nabla \cdot \boldsymbol{J}' + \frac{\partial}{\partial t}\rho' = 0. \tag{27c}$$

Here we place the term $\frac{\partial \boldsymbol{P}_s}{\partial t}$ in the current simply because it is produced by the movement of the medium boundary as triggered by mechanical force (see Fig. 1b), and the term $\rho_f\ \boldsymbol{v}$ is the current generated by the media movement acting on the free charges. Using the total displacement vector in Eq. (20), $\boldsymbol{D} = \boldsymbol{D}' + \boldsymbol{P}_s$, Eq. (26a–d) are stated as

$$\nabla \cdot \boldsymbol{D} = \rho_f \tag{28a}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{28b}$$

$$\nabla \times \boldsymbol{E} \cong -\frac{\partial}{\partial t}\boldsymbol{B} \tag{28c}$$

$$\nabla \times \boldsymbol{H} \cong \boldsymbol{J}_f + \rho_f \boldsymbol{v} + \frac{\partial}{\partial t}\boldsymbol{D} \tag{28d}$$

Eqs. (28a–d) can be simply referred as the *expanded* Maxwell's equations. The general solution of Eqs. (26a–d) or Eqs. (28a–d) are given in the section on Vector potential solution of the expanded Maxwell's equations.

For magnetic media

If the medium is a ferromagnetic material,

$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}) \tag{29}$$

where $\boldsymbol{B}$ is the magnetic field, $\boldsymbol{H}$ is the magnetizing field, $\boldsymbol{M}$ is magnetization, μ_0 is vacuum permeability. From Eq. (26d)

$$\nabla \times \boldsymbol{B} = \mu_0(\boldsymbol{J}_f + \rho_f \boldsymbol{v} + \frac{\partial}{\partial t}\boldsymbol{D}' + \frac{\partial}{\partial t}\boldsymbol{P}_s + \nabla \times \boldsymbol{M}) \tag{30}$$

Therefore, the displacement for magnetic medium would have three terms:

$$\boldsymbol{J}_{\mathrm{D}} = \varepsilon\frac{\partial}{\partial t}\boldsymbol{E} + \frac{\partial}{\partial t}\boldsymbol{P}_s + \nabla \times \boldsymbol{M} \tag{31}$$

Therefore, the displacement current may be expanded into three types: the current due to time variation of electric field rather than charge flow, which is responsible for the transmission of electromagnetic wave, first proposed by Maxwell; the passing-by flow of the charged medium boundaries due to external mechanical agitation, proposed by Wang; and the curl of the magnetization. Such definition in Eq. (31) expands the scope of the Maxwell's displacement current.

Conservation of energy as governed by the expanded Maxwell's equations

Starting from Eqs. (26a–d), we explore the energy conversion process as governed by the expanded Maxwell's equations. Using the mathematical identity

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) = \boldsymbol{H} \cdot (\nabla \times \boldsymbol{E}) - \boldsymbol{E} \cdot (\nabla \times \boldsymbol{H}) \tag{32}$$

and simply assuming $\boldsymbol{B} = \mu\boldsymbol{H}$ and $\boldsymbol{D}' = \varepsilon\boldsymbol{E}$ for general materials, using Eq. (26d), we have

$$\begin{aligned}\iiint_V (\boldsymbol{E} \cdot \boldsymbol{J}')d\boldsymbol{r} &= \iiint_V \left(\boldsymbol{E}\cdot[\nabla \times \boldsymbol{H} - \frac{\partial}{\partial t}\boldsymbol{D}']\right)d\boldsymbol{r} \\ &= \iiint_V \left(\boldsymbol{H}\cdot(\nabla\times\boldsymbol{E}) - \nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}) - \boldsymbol{E}\cdot\frac{\partial}{\partial t}\boldsymbol{D}'\right)d\boldsymbol{r} \\ &= -\iiint_V \left(\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}) + \boldsymbol{E}\cdot\frac{\partial \boldsymbol{D}'}{\partial t} + \boldsymbol{H}\cdot\frac{\partial \boldsymbol{B}}{\partial t}\right)d\boldsymbol{r} \\ &= -\oiint_S \boldsymbol{S}\cdot d\boldsymbol{s} - \iiint_V \left(\frac{\partial}{\partial t}u\right)d\boldsymbol{r}\end{aligned} \tag{33}$$

where $\boldsymbol{S}$ is the Poynting vector

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} \tag{34}$$

and u is the energy volume density of electromagnetic field

$$u = 1/2(\boldsymbol{B}\cdot\boldsymbol{H} + \boldsymbol{D}'\cdot\boldsymbol{E}) \tag{35}$$

We have

$$\iiint_V \left(\frac{\partial}{\partial t}u\right)d\boldsymbol{r} + \oiint_S \boldsymbol{S}\cdot d\boldsymbol{s} = -\iiint_V (\boldsymbol{E}\cdot\boldsymbol{J}')d\boldsymbol{r} \tag{36a}$$

$$\frac{\partial}{\partial t}u + \nabla\cdot\boldsymbol{S} = -\boldsymbol{E}\cdot\boldsymbol{J}' = -\boldsymbol{E}\cdot[\boldsymbol{J}_f + \rho_f\boldsymbol{v} + \frac{\partial}{\partial t}\boldsymbol{P}_s] \tag{36b}$$

This equation means that the increase of the internal electromagnetic field energy within a volume plus the rate of the radiated electromagnetic wave energy out of the volume surface is the negative of the rate of the energy done by the field on the external free current and the output current of the nanogenerator within the volume. This is the conservation of energy.

If we define a potential Φ as $\boldsymbol{E} = -\nabla\Phi$, using Eq. (26d),

$$\begin{aligned}\iiint_V (\boldsymbol{E}\cdot\boldsymbol{J}')\mathrm{d}\boldsymbol{r} &= -\iiint_V (\nabla\Phi\cdot\boldsymbol{J}')\mathrm{d}\boldsymbol{r} = -\iiint_V (\nabla\cdot(\Phi\boldsymbol{J}') - \Phi\nabla\cdot\boldsymbol{J}')\mathrm{d}\boldsymbol{r} \\ &= -\oiint_S \Phi\boldsymbol{J}'\cdot\mathrm{d}\boldsymbol{s} - \iiint_V \left(\Phi\frac{\partial}{\partial t}\rho'\right)\mathrm{d}\boldsymbol{r}\end{aligned} \tag{37}$$

Therefore, from Eq. (32) and Eq. (36), we have

$$\begin{aligned}\iiint_V \left(\frac{\partial}{\partial t}u\right)\mathrm{d}\boldsymbol{r} + \oiint_S \boldsymbol{S}\cdot d\boldsymbol{s} &= \oiint_S \Phi\left(\boldsymbol{J}_f + \rho_f\boldsymbol{v} + \frac{\partial}{\partial t}\boldsymbol{P}_s\right)\cdot\mathrm{d}\boldsymbol{s} \\ &+ \iiint_V \Phi\left(\frac{\partial}{\partial t}\rho_f - \frac{\partial}{\partial t}\nabla\cdot\boldsymbol{P}_s\right)\mathrm{d}\boldsymbol{r}\end{aligned} \tag{38}$$

The physical meaning is as follows: the increasing rate of the electromagnetic field energy within the volume plus the radiated electromagnetic wave energy out of the volume equal to the rate of energy input externally for driving the 'effective electric current' out of the volume surface and raising the electrostatic energy for the existing charges within the volume. In the first integral at the right-hand side of Eq. (38), the first term energy is supplied by an externally applied oscillating electric current (J_f) for generating electromagnetic wave, and the second term ($\rho_f\boldsymbol{v} + \frac{\partial}{\partial t}\mathbf{P}_s$) can be supplied by an external mechanical agitation (see Section on polarization term arising from mechanical triggering), which is the case for TENGs.

To fully understand the energy conversion process, we now use a pair of parallel electroplates capacitor as an example. If the surface electrostatic charge density on the plate surface is $\pm\sigma_s$, respectively, from the Gauss's law, the corresponding electric field between the plates is $E = \sigma_s/\varepsilon_0$. The voltage drop between the two electroplates is $V = Ex = x\ \sigma_s/\varepsilon_0$. The polarization vector from the electrostatic charges is $P_s = \varepsilon E = x\sigma_s$. The corre-

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sponding displacement current density is $A\frac{\partial P_s}{\partial t}=v\sigma_s A=qv$, where A is the area of the plate, and v is the velocity at which the two plates being separated by an external force. This is the process of inputting mechanical energy for power generation.

The polarization term $\mathbf{P}_s$ arising from mechanical triggering

If the electrostatic charges are assumed to be confined on the medium surface, as the case for nanogenerators, which is represented by a surface charge density function $\boldsymbol{\sigma}_s$ *($\boldsymbol{r}$,t)* on the surfaces of the media whose shape function is *f($\boldsymbol{r}$,t) = 0*, where the time is introduced to represent the instantaneous shape of the media with considering external mechanical triggering (Fig. 1), the equation for defining $\mathrm{P_s}$, can be expressed as [3]

$$\nabla\cdot\boldsymbol{P}_s=-\sigma_s(\boldsymbol{r},t)\delta(f(\boldsymbol{r},t)) \tag{39}$$

where $\delta(f(\boldsymbol{r},t))$ is a delta function that is introduced to confine the shape of the media $f(\boldsymbol{r},t)=0$ so that the polarization charges produced by non-electric field are confined on the medium surface, and which is defined as follows:

$$\delta(f(\boldsymbol{r},t))=\begin{cases}\infty & if\ f(\boldsymbol{r},t)=0\\ 0 & otherwise\end{cases} \tag{40a}$$

$$\int_{-\infty}^{\infty}\delta(f(\boldsymbol{r},t))dn=1 \tag{40b}$$

where $\boldsymbol{n}$ is the normal direction of the local surface, and dn is an integral along the surface normal direction of the media. It must be pointed out that Eq. (40a) may not be precisely mathematically, but it serves the purpose of indicating the charges are distributed on the surface. Such an inaccuracy is eliminated after converting Eq. (39) into its integral form as stated in Eq. (43). This is a simple treatment about the surface bound charges. It is important to note that the shapes of the dielectric media depend on time, because under external mechanical triggering, the shape and distribution of the dielectric media can vary, which is the reason for introducing the time t in $f(\boldsymbol{r},t)$. The potential produced by the surface electrostatic charges results in a redistribution of free charges in the electrodes in order to satisfy the boundary conditions across media boundary. For metal electrodes, the surface has to maintain a constant potential at low frequency. The total potential in space is that both by free charges and the surface electrostatic charges, and the total potential distribution in space is given by

$$\varepsilon\nabla^2\Phi=-\rho_f-\boldsymbol{\sigma}_s(\boldsymbol{r},t)\delta(f(\boldsymbol{r},t)) \tag{41}$$

The general solution of Φ is made of two components: a homogenous solution Φ_0, which satisfies the Laplace equation: $\nabla^2\Phi_0=0$; and a special solution Φ_s. Φ needs to satisfy all of the boundary conditions.

As for nanogenerators, our main concern here is the displacement current. If we define a "potential" induced by $\mathrm{P_s}$ by:

$$\mathrm{P_s}=-\nabla\varphi_s(\mathrm{r},t), \tag{42a}$$

we have

$$\nabla^2\varphi_s(\boldsymbol{r},t)=\sigma_s(\boldsymbol{r},t)\delta(f(\boldsymbol{r},t)) \tag{42b}$$

The general solution of φ_s is made of two components: a homogenous solution φ_0, which satisfies the Laplace equation: $\nabla^2\varphi_0=0$; and a special solution φ_{s0}. φ_s needs to satisfy all of the boundary conditions. Now let's look at the special solution:

$$\varphi_{s0}(\boldsymbol{r},\ t)=\frac{1}{4\pi}\oiint_S\frac{\boldsymbol{\sigma}_s(\boldsymbol{r}',\ t)}{|\boldsymbol{r}\ -\ \boldsymbol{r}'|}\boldsymbol{ds}' \tag{43}$$

where d$\boldsymbol{s}'$ is an integral over the surface of the dielectric media (Fig. 2a). Therefore, the polarization arising from the surface charge density is

$$\boldsymbol{P}_s=-\nabla\varphi_{s0}(\boldsymbol{r},t)=\frac{1}{4\pi}\oiint_S\boldsymbol{\sigma}_s(\boldsymbol{r}',t')\frac{\boldsymbol{r}\ -\ \boldsymbol{r}'}{|\boldsymbol{r}\ -\ \boldsymbol{r}'|^3}d\boldsymbol{s}' \tag{44}$$

$$\begin{aligned}\frac{\partial}{\partial t}\boldsymbol{P}_s&=\frac{1}{4\pi}\frac{\partial}{\partial t}\oiint_S\left\{\boldsymbol{\sigma}_s(\boldsymbol{r}',\ t)\frac{\boldsymbol{r}\ -\ \boldsymbol{r}'}{|\boldsymbol{r}\ -\ \boldsymbol{r}'|^3}\ d\boldsymbol{s}'\right\}\\&=\frac{1}{4\pi}\oiint_S\frac{\partial}{\partial t}\boldsymbol{\sigma}_s(\boldsymbol{r}',\ t)\frac{\boldsymbol{r}\ -\ \boldsymbol{r}'}{|\boldsymbol{r}\ -\ \boldsymbol{r}'|^3}\ d\boldsymbol{s}'\\&\quad+\frac{1}{4\pi}\oiint_S(\boldsymbol{v}\cdot\nabla')\left[\boldsymbol{\sigma}_s(\boldsymbol{r}',\ t)\frac{\boldsymbol{r}\ -\ \boldsymbol{r}'}{|\boldsymbol{r}\ -\ \boldsymbol{r}'|^3}\right]d\boldsymbol{s}'\end{aligned} \tag{45}$$

where ∇' is the Laplace operator applied to r'. In Eq. (45), the first term is the contribution made by the time variation of the surface charge density, and the second term is related to the movement speed $\boldsymbol{v}$ of the medium, which is the result of mechanical energy input, and it is the key term for the current output of the nanogenerators. Numerical calculation using for quantifying TENG performance has been carried out [27,28]. Here, we could generalize the result in Eq. (45) into cases in which there is a variation of the medium movement velocity $\boldsymbol{v}$ across the medium volume and surface.

Electrostatic approximation – what is missing?

We now consider a case if we only consider the contribution of electrostatic charges on the medium boundary to the distribution of electric field, which means that the corresponding Maxwell's equations are:

$$\nabla\cdot\boldsymbol{D}'=\rho_f-\nabla\cdot\boldsymbol{P}_s \tag{46a}$$

$$\nabla\cdot\boldsymbol{B}=0 \tag{46b}$$

$$\nabla\times\boldsymbol{E}=-\frac{\partial}{\partial t}\boldsymbol{B} \tag{46c}$$

$$\nabla\times\boldsymbol{H}=\boldsymbol{J}_f+\rho_f\boldsymbol{v}+\frac{\partial}{\partial t}\boldsymbol{D}' \tag{46d}$$

where there is no change in the Ampere-Maxwell equation. In such a case, using Eq. (16b), we have

$$\begin{aligned}\nabla\cdot(\nabla\times\boldsymbol{H})&=\nabla\cdot(\boldsymbol{J}_f+\rho_f\boldsymbol{v})+\frac{\partial}{\partial t}\nabla\cdot\boldsymbol{D}'\\&=\nabla\cdot(\boldsymbol{J}_f+\rho_f\boldsymbol{v})+\frac{\partial}{\partial t}\rho_f-\nabla\cdot\frac{\partial}{\partial t}\boldsymbol{P}_s=-\nabla\cdot\frac{\partial}{\partial t}\boldsymbol{P}_s\end{aligned} \tag{47}$$

Mathematically, $\nabla\cdot(\nabla\times\boldsymbol{H})=0$, which requires that $\nabla\cdot\frac{\partial}{\partial t}\boldsymbol{P}_s=0$ in order to satisfy the law of charge conservation. We know that $\nabla\cdot\frac{\partial}{\partial t}\boldsymbol{P}_s\neq0$, therefore, the inclusion of only the electrostatic charge component $(-\nabla\cdot\boldsymbol{P}_s)$ in the Gauss law, but missing the displacement current density caused by the moving medium boundary $(\frac{\partial}{\partial t}\boldsymbol{P}_s)$ in the Ampere-Maxwell's law makes the Maxwell's equations not fully consistent with the law of charge conservation. This means that our introduction of the $\boldsymbol{P}_s$ term in the displacement field $\boldsymbol{D}$ (Eq. (20)) is the most logic approach for deal with the problem of moving charged boundaries in electrodynamics [3].

Now let's answer the question that why the term $\frac{\partial}{\partial t}\boldsymbol{P}_s$ was missed in the original Maxwell's equations. From our derivation above, the exclusive condition under which the differential form of the Maxwell's equations was derived is that the medium volume, boundary and distribution are fixed without change over time. Therefore, the exclusive focus of Maxwell was developing

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the theory of electromagnetic waves. Now by introducing the effect of the mechanical energy triggering on the medium and even the surface charges, an additional term is required. Therefore, the definition of the displacement current by Maxwell only apply to the case of electromagnetic waves. The added term $\frac{\partial}{\partial t}\boldsymbol{P}_s$ accounts for the fundamental of nanogenerators. Both could be decoupled due to large differences in frequency, but with the increase of mechanical triggering frequency, both terms could be coupled, which are comprehensively included in the expanded Maxwell's equations. Fig. 6 presents a summary of above discussion.

Vector potential solution of the expanded Maxwell's equations

We now look into the solutions of Eqs. (26a–d). The $\boldsymbol{E}$ and $\boldsymbol{B}$ can be calculated by introducing the vector magnetic potential $\boldsymbol{A}$:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{48}$$

and the scalar electric potential φ for electrostatics, we define

$$\boldsymbol{E} = -\nabla\varphi - \frac{\partial \boldsymbol{A}}{\partial t} \tag{49}$$

Substitute Eqs. (48) and (49) into Eqs. (26a–d) and make use of the constitutive relations, we have,

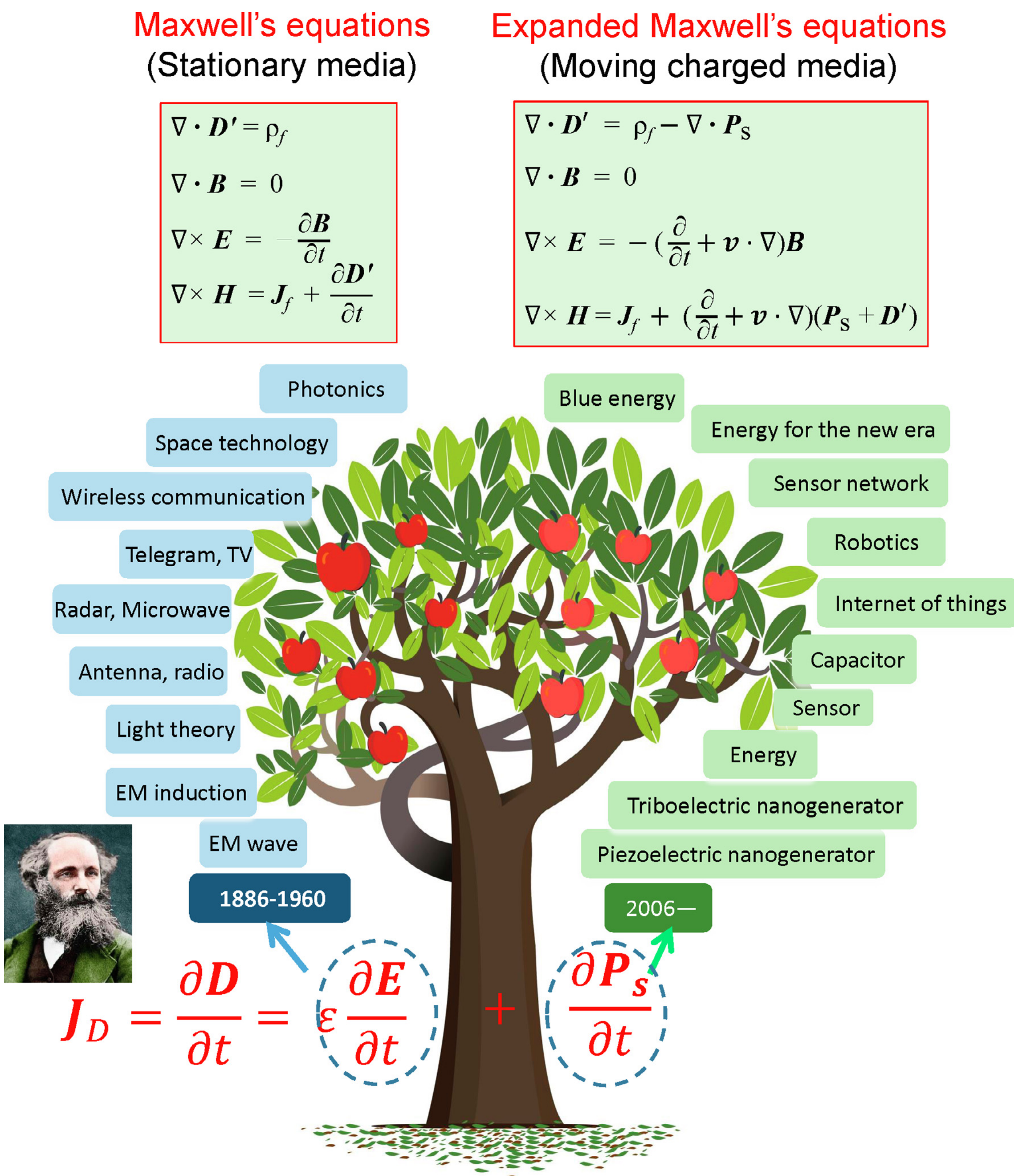

FIGURE 6

A comparison of the Maxwell's equations for stationary media and moving charged media. Schematic diagram showing the contribution of the displacement current first proposed by Maxwell as a term of time-variation of electric field and how it contributes to the development of electromagnetic field theory. The term $\frac{\partial \boldsymbol{P}_s}{\partial t}$ in the expanded Maxwell equation introduced by Wang is the foundation of TENG, which is called the Wang term.

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$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu \boldsymbol{J}' + \nabla\left(\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t}\right) \tag{50}$$

where $c = \frac{1}{\sqrt{\mu\varepsilon}}$ is the speed of light in the medium. Using Lorentz gauge,

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0 \tag{51}$$

which makes the second term on the right-hand side of Eq. (50) vanish, so we obtain

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu \boldsymbol{J}' \tag{52}$$

This is a nonhomogeneous wave equation for vector potential $\boldsymbol{A}$. It is called a wave equation because its solutions represent waves traveling with a velocity equal to c.

A corresponding wave equation for the scalar potential φ can be obtained by substituting Eq. (49) in Eq. (26c), we have

$$\nabla^2 \varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = -\frac{\rho'}{\varepsilon} \tag{53}$$

which is a nonhomogeneous wave equation for scalar potential φ. Once the solution of $\boldsymbol{A}$ and φ can be found, the total electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ can be calculated.

The solutions of $\boldsymbol{A}$ and φ are each made of two components: homogeneous component that satisfy

$$\nabla^2 \boldsymbol{A}_h - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{A}_h = 0 \tag{54}$$

$$\nabla^2 \varphi_h - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi_h = 0 \tag{55}$$

And the special solutions that satisfy Eqs. (54) and (55). The total solutions of potential $\boldsymbol{A}$ and φ should satisfy the boundary conditions for both $\boldsymbol{B}$ and $\boldsymbol{E}$. By using the Green function, the special solutions for Eqs. (54) and (55) are given by (See Chapter 6 in ref. [1] for details):

$$\varphi_s(\boldsymbol{r}, t) = \frac{1}{4\pi\varepsilon}\iiint_V \frac{\rho'(\boldsymbol{r}', t')}{|\boldsymbol{r} - \boldsymbol{r}'|} d\boldsymbol{r}' \tag{56}$$

and

$$\boldsymbol{A_s}(\boldsymbol{r}, t) = \frac{\mu}{4\pi}\iiint_V \frac{\boldsymbol{J}'(\boldsymbol{r}', t')}{|\boldsymbol{r} - \boldsymbol{r}'|} d\boldsymbol{r}' \tag{57}$$

where t' is the retardation time $t' = t - \left|\frac{\boldsymbol{r}-\boldsymbol{r}'}{c}\right|$, and c is the speed of light. Substituting Eqs. (56) and (57) into Eqs. (48) and (49) and through some mathematical derivations, we have the electromagnetic wave in free space as:

$$\boldsymbol{E_s}(\boldsymbol{r}, t) = \frac{1}{4\pi\varepsilon}\iiint_V \left[\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\rho'(\boldsymbol{r}', t') + \frac{1}{c}\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^2}\frac{\partial \rho'(\boldsymbol{r}', t')}{\partial t'} - \frac{1}{c^2|\boldsymbol{r}-\boldsymbol{r}'|}\frac{\partial \boldsymbol{J}'(\boldsymbol{r}', t')}{\partial t'}\right] d\boldsymbol{r}' \tag{58}$$

$$\boldsymbol{B_s}(\boldsymbol{r}, t) = \frac{\mu}{4\pi}\iiint_V \left[\boldsymbol{J}'(\boldsymbol{r}', t') \times \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\rho'(\boldsymbol{r}', t') + \frac{1}{c}\frac{\partial \boldsymbol{J}'(\boldsymbol{r}', t')}{\partial t'} \times \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^2}\right] d\boldsymbol{r}' \tag{59}$$

The solutions in Eqs. (58) and (59) are the electromagnetic waves in free space if there is no dielectric media with boundary. In a case there is media boundaries, one must consider the full solution of the equations with considering the satisfactions of boundary conditions. The detailed calculation of Eqs. (56) and (57) are given in Ref. (5) for the four modes of TENG.

Strategies on the solutions of the expanded Maxwell's equations

We now present the solution of the fully expanded Maxwell's equations without ignoring the speed dependent terms. Equations (15a-d) are rewritten as:

$$\nabla \cdot \boldsymbol{D}' = \rho_f \tag{60a}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{60b}$$

$$\nabla \times \boldsymbol{E} = -\frac{D}{Dt}\boldsymbol{B} \tag{60c}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_f + \frac{D}{Dt}\boldsymbol{D}' \tag{60d}$$

The operator represents the effect of the translation of the origin $\boldsymbol{r}_0$(t) of the coordination system for the moving media in its stationary/-fixed reference frame $\boldsymbol{r}_v$ relative to the original reference frame $\boldsymbol{r}$ of the entire system: $\boldsymbol{r} = \boldsymbol{r}_0(t) + \boldsymbol{r}_v$ (see Fig. 2b) for definition), on the time differentiation, which can be mathematically expressed as follows:

$$\frac{d}{dt}F(\boldsymbol{r}, t) = \frac{\partial}{\partial t}F(\boldsymbol{r}, t) + \frac{\partial \boldsymbol{r}}{\partial t}\cdot \nabla F(\boldsymbol{r}, t) = \frac{\partial}{\partial t}F(\boldsymbol{r}, t) + \boldsymbol{v}\cdot \nabla F(\boldsymbol{r}, t) = \frac{D}{Dt}F(\boldsymbol{r}, t) \tag{61}$$

It can be proved that the operator satisfied the commutation rule: $\nabla\frac{D}{Dt} = \frac{D}{Dt}\nabla$. We now introduce a new vector magnetic potential, $\boldsymbol{A}_v$:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}_v \tag{62a}$$

and a new scalar electric potential φ_v for electrostatics, we define

$$\boldsymbol{E} = -\nabla\varphi_v - \frac{D}{Dt}\boldsymbol{A}_v \tag{62b}$$

Substitute Eqs. (62a, b) into Eqs. (60a–d) and make use of the constitutive relations, we have,

$$\nabla^2 \boldsymbol{A}_v - \frac{1}{c^2}\frac{D^2}{Dt^2}\boldsymbol{A}_v = -\mu \boldsymbol{J}_f \tag{63}$$

$$\nabla^2 \varphi_v - \frac{1}{c^2}\frac{D^2}{Dt^2}\varphi_v = -\frac{\rho_f}{\varepsilon} \tag{64}$$

where $\frac{D^2}{Dt^2} = \left[\frac{\partial}{\partial t} + (\boldsymbol{v}\cdot\nabla)\right]\left[\frac{\partial}{\partial t} + (\boldsymbol{v}\cdot\nabla)\right] = \frac{\partial^2}{\partial t^2} + 2(\boldsymbol{v}\cdot\nabla)\frac{\partial}{\partial t} + (\boldsymbol{v}\cdot\nabla)(\boldsymbol{v}\cdot\nabla)$.and the Lorentz gauge must be satisfied:

$$\nabla \cdot \boldsymbol{A}_v + \frac{1}{c^2}\frac{D}{Dt}\varphi_v = 0 \tag{65}$$

These are nonhomogeneous wave equations for vector potential $\boldsymbol{A}_v$ and φ_v which are non-linear differential equations. The total solutions may have to be solved numerically, and the total solutions must satisfy the boundary conditions as defined in Eqs. (19a–d).

Now for the case with the inclusion of the surface electrostatic charges, as discussed in Section on Expanded Maxwell's equations for moving charged media, by a substitution of $\boldsymbol{D}'$ by $\boldsymbol{D} = \boldsymbol{D}' + \boldsymbol{P}_s$ in Eqs. (60a–d), we have

$$\nabla \cdot \boldsymbol{D}' = \rho'' \tag{66a}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{66b}$$

$$\nabla \times \boldsymbol{E} = -\frac{D}{Dt}\boldsymbol{B} \tag{66c}$$

$$\nabla\times\boldsymbol{H}=\boldsymbol{J}''+\frac{D}{Dt}\boldsymbol{D}' \tag{66d}$$

where:

$$\rho''=\rho_f-\nabla\cdot\boldsymbol{P}_s \tag{66e}$$

$$\boldsymbol{J}''=\boldsymbol{J}_f+\frac{D}{Dt}\boldsymbol{P}_s=\boldsymbol{J}_f+(\boldsymbol{v}\cdot\nabla)\boldsymbol{P}_s+\frac{\partial}{\partial t}\boldsymbol{P}_s \tag{66f}$$

The conservation of charges is satisfied:

$$\nabla\cdot\boldsymbol{J}''+\frac{D}{Dt}\rho''=0, \tag{66g}$$

The vector potential and scalar potential solutions of Eq. (62a, b) can also be received from Eqs. (64) and (65) except one has to replace $\boldsymbol{J}_f$ by $[\boldsymbol{J}_f+(\boldsymbol{v}\cdot\nabla)\boldsymbol{P}_s+\frac{\partial}{\partial t}\boldsymbol{P}_s=\boldsymbol{J}_f+\frac{D}{Dt}\boldsymbol{P}_s]$, and ρ_f by $[\rho_f-\nabla\cdot\boldsymbol{P}_s]$. In Eq. (66d), the additional current related to the moving media is given by the second and third term: $(\boldsymbol{v}\cdot\nabla)\boldsymbol{P}_s+\frac{\partial}{\partial t}\boldsymbol{P}_s$. Therefore, for low-moving speed medium, the $\frac{\partial}{\partial t}\boldsymbol{P}_s$ is the dominant contribution. This is the fundamental output current of the TENG at short circuit.

We now consider the practical calculations as for the case presented in Fig. 2, in which different media could move at different speed, but the observer is at the origin of the reference frame for media A. The governing equations inside stationary medium A of dielectric permittivity ε_A are [by take $\boldsymbol{v}=0$ in Eqs. (22a–d)]:

$$\varepsilon_A\nabla\cdot\boldsymbol{E}=\rho_f-\nabla\cdot\boldsymbol{P}_s \tag{67a}$$

$$\nabla\cdot\boldsymbol{B}=0 \tag{67b}$$

$$\nabla\times\boldsymbol{E}=-\frac{\partial}{\partial t}\boldsymbol{B} \tag{67c}$$

$$\nabla\times\boldsymbol{H}=\boldsymbol{J}_f+\frac{\partial}{\partial t}\boldsymbol{P}_s+\varepsilon_A\frac{\partial}{\partial t}\boldsymbol{E} \tag{67d}$$

The boundary conditions are given by Eqs. (19a–d) for $\boldsymbol{v}=0$. The strategy for the solutions is given in Section on Vector potential solution of the expanded Maxwell's equations.

The governing equations inside the moving medium B (Fig. 2b) that is translating at a speed $\boldsymbol{v}_B$ and has a dielectric permittivity ε_B are:

$$\varepsilon_B\nabla\cdot\boldsymbol{E}=\rho_f-\nabla\cdot\boldsymbol{P}_s \tag{68a}$$

$$\nabla\cdot\boldsymbol{B}=0 \tag{68b}$$

$$\nabla\times\boldsymbol{E}=-\frac{D}{Dt}\boldsymbol{B} \tag{68c}$$

$$\nabla\times\boldsymbol{H}=\boldsymbol{J}_f+(\boldsymbol{v}_B\cdot\nabla)\boldsymbol{P}_s+\frac{\partial}{\partial t}\boldsymbol{P}_s+\varepsilon_B\frac{D}{Dt}\boldsymbol{E} \tag{68d}$$

The boundary conditions are given by Eq. (19a–d). The solutions follow the strategy presented in Eqs. (62–65). If one introduce another medium C that moves at speed $\boldsymbol{v}_C$, Eqs. (68a–d) can be equivalently applied.

The governing equations in vacuum between media A and B are:

$$\varepsilon_0\nabla\cdot\boldsymbol{E}=0 \tag{69a}$$

$$\nabla\cdot\boldsymbol{B}=0 \tag{69b}$$

$$\nabla\times\boldsymbol{E}=-\frac{\partial}{\partial t}\boldsymbol{B} \tag{69c}$$

$$\nabla\times\boldsymbol{H}=\varepsilon_0\frac{\partial}{\partial t}\boldsymbol{E} \tag{69d}$$

The strategy for the solutions is given in Section on Vector potential solution of the expanded Maxwell's equations. Eqs. (67–69) form the theoretical framework for describing the electromagnetic behavior of a moving media system.

Solutions of the vector and scalar potentials

The perturbation theory

We now use perturbation theory to solve Eqs. (63–65) by expanding them in the order of $\boldsymbol{v}=v\xi$, where ξ is the unit vector representing the direction of $\boldsymbol{v}$. We can have following expansions:

$$\boldsymbol{A}_v=\boldsymbol{a}_0+\beta\boldsymbol{a}_1+\beta^2\boldsymbol{a}_2+\cdots \tag{70a}$$

$$\varphi_v=\Phi_0+\beta\Phi_1+\beta^2\Phi_2+\cdots \tag{70b}$$

where $\beta=v/c$. Substituting Eqs. (70a)-(70b) into Eqs. (63–65), the corresponding equations for the same order of β are:

Zeroth order:

$$\nabla^2\boldsymbol{a}_0-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{a}_0=-\mu\boldsymbol{J}_f \tag{71a}$$

$$\nabla^2\Phi_0-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi_0=-\frac{\rho_f}{\varepsilon} \tag{71b}$$

$$\nabla\cdot\boldsymbol{a}_0+\frac{1}{c^2}\frac{\partial}{\partial t}\Phi_0=0 \tag{71c}$$

First order:

$$\nabla^2\boldsymbol{a}_1-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{a}_1=\frac{2}{c}(\xi\cdot\nabla)\frac{\partial}{\partial t}\boldsymbol{a}_0 \tag{72a}$$

$$\nabla^2\Phi_1-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi_1=\frac{2}{c}(\xi\cdot\nabla)\frac{\partial}{\partial t}\Phi_0 \tag{72b}$$

$$\nabla\cdot\boldsymbol{a}_1+\frac{1}{c^2}\frac{\partial}{\partial t}\Phi_1=-\frac{1}{c}(\xi\cdot\nabla)\frac{\partial}{\partial t}\Phi_0 \tag{72c}$$

Second order:

$$\nabla^2\boldsymbol{a}_2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{a}_2=\frac{2}{c}(\xi\cdot\nabla)\frac{\partial}{\partial t}\boldsymbol{a}_1+(\xi\cdot\nabla)(\xi\cdot\nabla)\boldsymbol{a}_0 \tag{73a}$$

$$\nabla^2\Phi_2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi_2=\frac{2}{c}(\xi\cdot\nabla)\frac{\partial}{\partial t}\Phi_1+(\xi\cdot\nabla)(\xi\cdot\nabla)\Phi_0 \tag{73b}$$

$$\nabla\cdot\boldsymbol{a}_2+\frac{1}{c^2}\frac{\partial}{\partial t}\Phi_2=-\frac{1}{c}(\xi\cdot\nabla)\frac{\partial}{\partial t}\Phi_1 \tag{73c}$$

The higher orders can be calculated as well. The solutions can be derived step by step using the method presented in Section on Vector potential solution of the expanded Maxwell's equations. But the total solution needs to satisfy the boundary conditions.

The iteration method

In addition, the iteration method can be adopted for solving the special solution of Eqs. (63) and (64). By using the full expansion

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of the operator $\frac{D^2}{Dt^2}$, Eqs. (63–64) are expressed as:

$$\nabla^2 \boldsymbol{A}_v - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{A}_v = -\mu \boldsymbol{J}_f + \frac{1}{c^2}[2(\boldsymbol{v}\cdot\nabla)\frac{\partial}{\partial t} + (\boldsymbol{v}\cdot\nabla)(\boldsymbol{v}\cdot\nabla)]\boldsymbol{A}_v \quad (74a)$$

$$\nabla^2 \varphi_v - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi_v = -\frac{\rho_f}{\varepsilon} + \frac{1}{c^2}[2(\boldsymbol{v}\cdot\nabla)\frac{\partial}{\partial t} + (\boldsymbol{v}\cdot\nabla)(\boldsymbol{v}\cdot\nabla)]\varphi_v \quad (74b)$$

Besides the homogenous solutions that satisfy Eqs. (54–55), the special solutions for the inhomogeneous parts of Eqs. (74a–b) are thus:

$$\boldsymbol{A}_v(\boldsymbol{r},t) = \frac{\mu}{4\pi}\iiint_V \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{J}_f(\boldsymbol{r}',\ t')\,d\boldsymbol{r}' - \frac{1}{4\pi c^2}\iiint_V \times \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\left\{\left[2(\boldsymbol{v}\cdot\nabla')\frac{\partial}{\partial t'} + (\boldsymbol{v}\cdot\nabla')(\boldsymbol{v}\cdot\nabla')\right]\boldsymbol{A}_v(\boldsymbol{r}',\ t')\right\}d\boldsymbol{r}' \quad (75a)$$

$$\varphi_v(\boldsymbol{r},\ t) = \frac{1}{4\pi\varepsilon}\iiint_V \frac{1}{|\boldsymbol{r}\ -\ \boldsymbol{r}'|}\rho_f(\boldsymbol{r}',\ t')\,d\boldsymbol{r}' - \frac{1}{4\pi c^2}\iiint_V \times \frac{1}{|\boldsymbol{r}\ -\ \boldsymbol{r}'|}\left\{\left[2(\boldsymbol{v}\cdot\nabla')\frac{\partial}{\partial t'} + (\boldsymbol{v}\cdot\nabla')(\boldsymbol{v}\cdot\nabla')\right]\varphi v(\boldsymbol{r}',\ t')\right\}d\boldsymbol{r}' \quad (75b)$$

These are integral equations, the solutions of which can be calculated step by step in the order of v by iteration method. Taking Eq. (75a) as an example. The first term containing $\boldsymbol{J}_f(\boldsymbol{r}', t')$ in the integral can be treated as the zeroth order solution, using which the first order solution can be derived by replacing the $\boldsymbol{A}_v$ in the second term by the zeroth order solution. Then substitute the $\boldsymbol{A}_v$ in the integral by the first order solution to receive the second order solution, etc.

Solution of the expanded Maxwell's equations in frequency space

In general, the dielectric permittivity is frequency dependent, rather than a constant. To include the frequency in the entire theory, we use the Fourier transform and inverse Fourier transform in time and frequency space as defined by:

$$a(\boldsymbol{r},\ \omega) = \int_{-\infty}^{\infty} dt\ e^{i\omega t}\ a(\boldsymbol{r},\ t) \quad (76a)$$

$$a(\boldsymbol{r},\ \omega) = a(\boldsymbol{r},\ t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ e^{-i\omega t}\ a(\boldsymbol{r},\ w) \quad (76b)$$

The purpose of introducing frequency space is to simplify the relationship between the displacement field $\boldsymbol{D}'$ and electric field $\mathbf{E}$, magnetic field $\mathbf{H}$ and magnetic flux density $\mathbf{B}$ as follows:

$$\boldsymbol{D}'(\boldsymbol{r},\omega) = \varepsilon(\omega)\boldsymbol{E}\ (\boldsymbol{r},\omega) \quad (77a)$$

$$\boldsymbol{B}(\boldsymbol{r},\omega) = \mu(\omega)\boldsymbol{H}\ (\boldsymbol{r},\omega) \quad (77b)$$

Otherwise, $\boldsymbol{D}'$ is a convolution of ε and,$\boldsymbol{E}$ in time space. Note, we use the same symbols to represent the real space and reciprocal space except the variables.

Using the Fourier transform and in frequency space, Eqs. (66a–d) become:

$$\varepsilon(\omega)\ \nabla\cdot\boldsymbol{E}(\boldsymbol{r},\omega) = \rho''(\boldsymbol{r},\omega) \quad (78a)$$

$$\mu(\omega)\ \nabla\cdot\boldsymbol{H}(\boldsymbol{r},\omega) = 0 \quad (78b)$$

$$\nabla\times\boldsymbol{E}(\boldsymbol{r},\omega) = -\mu(\omega)\ \frac{D}{D\xi}\boldsymbol{H}(\boldsymbol{r},\omega) \quad (78c)$$

$$\nabla\times\boldsymbol{H}(\boldsymbol{r},\omega) = \boldsymbol{J}''(\boldsymbol{r},\omega) + \varepsilon(\omega)\frac{D}{D\xi}\boldsymbol{E}(\boldsymbol{r},\omega) \quad (78d)$$

where

$$\frac{D}{D\xi} = -i\omega + \boldsymbol{v}\cdot\nabla \quad (78e)$$

By applying a curl operator $\nabla\times$ on Eq. (78c) and use Eqs. (78b) and (78d), we have

$$\nabla^2\boldsymbol{E}(\boldsymbol{r},\omega) - \mu\varepsilon\frac{D^2}{D\xi^2}\boldsymbol{E}(\boldsymbol{r},\omega) = \mu\frac{D}{D\xi}\boldsymbol{J}''(\boldsymbol{r},\omega) + \nabla\rho''(\boldsymbol{r},\omega)/\varepsilon \quad (79a)$$

where

$$\frac{D^2}{D\xi^2} = (-i\omega + \boldsymbol{v}\cdot\nabla)(-i\omega + \boldsymbol{v}\cdot\nabla) \quad (79b)$$

By applying a curl operator $\nabla\times$ on Eq. (78d) and use Eqs. (7bb) and (7bc), we have

$$\nabla^2\boldsymbol{H}(\boldsymbol{r},\omega) - \mu\varepsilon\frac{D^2}{D\xi^2}\boldsymbol{H}(\boldsymbol{r},\omega) = \nabla\times\boldsymbol{J}''(\boldsymbol{r},\omega) \quad (79c)$$

Equations (79a–c) are the standard equations that governs the distribution of electromagnetic waves at a specific frequency in space for a moving media, the solution of which could be complex because of the involvement of the operators in the equations.

The Hertz vector method

We now introduce the method of using Hertz vector for solving the expanded Maxwell's equations. To make the mathematical simple, we ignore the $\boldsymbol{v}$ dependent terms at the left-hand side, Eqs. (26a–d)) can be transformed in the frequency space by time Fourier transform as:

$$\nabla\cdot\boldsymbol{D}'(\boldsymbol{r},\omega) = \rho'(\boldsymbol{r},\omega) \quad (80a)$$

$$\nabla\cdot\boldsymbol{B}(\boldsymbol{r},\omega) = 0 \quad (80b)$$

$$\nabla\times\boldsymbol{E}(\boldsymbol{r},\omega) = i\omega\boldsymbol{B}(\boldsymbol{r},\omega) \quad (80c)$$

$$\nabla\times\boldsymbol{H}(\boldsymbol{r},\omega) = \boldsymbol{J}'(\boldsymbol{r},\omega) - i\omega\boldsymbol{D}'(\boldsymbol{r},\omega) \quad (80d)$$

We now use the Hertz vector $\mathbf{\Pi}(\boldsymbol{r},\omega)$ to reformulate the Maxwell's equations [29]. By defining,

$$\boldsymbol{E}(\boldsymbol{r},\omega) = \nabla[\nabla\cdot\mathbf{\Pi}(\boldsymbol{r},\omega)] + \mu\varepsilon\omega^2\mathbf{\Pi}(\boldsymbol{r},\omega) \quad (81a)$$

$$\boldsymbol{H}(\boldsymbol{r},\omega) = -i\omega\varepsilon\ \nabla\times\mathbf{\Pi}(\boldsymbol{r},\omega) \quad (81b)$$

Substitute Eqs. (81a, b) to Eq. (80d),

$$\nabla\times\boldsymbol{H}(\boldsymbol{r},\omega) = -i\omega\varepsilon\ \nabla\times[\nabla\times\mathbf{\Pi}] = -i\omega\varepsilon[\nabla(\nabla\cdot\mathbf{\Pi}) - \nabla^2\mathbf{\Pi}] = \boldsymbol{J}' - i\omega\varepsilon[\nabla(\nabla\cdot\mathbf{\Pi}) + \mu\varepsilon\omega^2\mathbf{\Pi}] \quad (82)$$

So, we have:

$$(\nabla^2 + \omega^2\varepsilon(\omega)\mu(\omega))\mathbf{\Pi}(\boldsymbol{r},\omega) = \frac{\boldsymbol{J}'(\boldsymbol{r},\omega)}{i\omega\varepsilon} \quad (83)$$

It can be proved that Eqs. (80a–c) are automatically satisfied with the use of Eq. (83) and charge conservation law (Eq. (27c)):

$$\nabla\cdot\boldsymbol{J}'(\boldsymbol{r},\omega) - i\omega\rho'(\boldsymbol{r},\omega) = 0 \quad (84)$$

The full solution of Hertz vector has two components: homogeneous solution that is determined by:

$$(\nabla^2 + \omega^2\mu\varepsilon)\mathbf{\Pi}_h(\boldsymbol{r},\omega) = 0 \quad (85)$$

And a special solution that satisfies:

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$$(\nabla^2 + \omega^2\mu\varepsilon)\mathbf{\Pi}_s(\boldsymbol{r},\omega) = \frac{\boldsymbol{J}'(\boldsymbol{r},\omega)}{i\omega\varepsilon} \tag{86}$$

The special solution Π_s can be derived using Green's function,

$$\mathbf{\Pi}_s(\boldsymbol{r},\omega) = -\frac{1}{4\pi i\omega\varepsilon}\iiint_V \frac{exp[i\omega\sqrt{\mu\varepsilon}|\boldsymbol{r}-\boldsymbol{r}'|]}{|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{J}'(\boldsymbol{r}',\omega)d\boldsymbol{r}' \tag{87}$$

The full solution is received by matching the boundary conditions for $\boldsymbol{E}$ and $\boldsymbol{B}$. The introduction of frequency dependence of dielectric permittivity is important to cover the dispersion of the media. Such an extension is required especially at higher frequencies.

We now expand the definition of the Hertz vector $\Pi(\boldsymbol{r},\omega)$ to solve the fully expanded Maxwell Equations (78a–d). By defining:

$$\boldsymbol{E}(\boldsymbol{r},\ \omega) = \nabla[\nabla\cdot\Pi(\boldsymbol{r},\omega)] - \mu\varepsilon\frac{D^2}{D\xi^2}\Pi(\boldsymbol{r},\omega) \tag{88a}$$

$$\boldsymbol{H}(\boldsymbol{r},\omega) = \varepsilon\frac{D}{D\xi}\nabla\times\Pi(\boldsymbol{r},\omega) \tag{88b}$$

and define:

$$\mathbf{\Xi}(\boldsymbol{r},\ \omega) = \frac{D}{D\xi}\Pi(\boldsymbol{r},\omega) = (-i\omega + \boldsymbol{v}\cdot\nabla)\Pi(\boldsymbol{r},\omega) \tag{88c}$$

substituting Eqs. (88a, b) into Eq. (78d), we have

$$\begin{aligned}\nabla\times\boldsymbol{H}(\boldsymbol{r},\omega) &= \varepsilon\nabla\times[\nabla\times\mathbf{\Xi}(\boldsymbol{r},\omega)]\\ &= \varepsilon[\nabla(\nabla\cdot\mathbf{\Xi}(\boldsymbol{r},\omega)) - \nabla^2\mathbf{\Xi}(\boldsymbol{r},\omega)]\\ &= \boldsymbol{J}''(\boldsymbol{r},\omega) + \varepsilon[\nabla(\nabla\cdot\mathbf{\Xi}(\boldsymbol{r},\omega)) - \mu\varepsilon\frac{D^2}{D\xi^2}\mathbf{\Xi}(\boldsymbol{r},\omega)]\end{aligned} \tag{89}$$

Therefore:

$$\left(\nabla^2 - \mu\varepsilon\frac{D^2}{D\xi^2}\right)\mathbf{\Xi}(\boldsymbol{r},\omega) = -\frac{\boldsymbol{J}'(\boldsymbol{r},\omega)}{\varepsilon} \tag{90}$$

It can be proved that Eqs. (88a–c) are automatically satisfied with the use of Eq. (90) and charge conservation law:

$$\nabla\cdot\boldsymbol{J}'(\boldsymbol{r},\omega) + \frac{D}{D\xi}\rho'(\boldsymbol{r},\omega) = 0 \tag{91}$$

For a simple case if there is no space current $\boldsymbol{J}' = 0$, using the Fourier form of the Hertz vector:

$$\mathbf{\Xi}(\boldsymbol{r},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\boldsymbol{k}\hat{A}\cdot\boldsymbol{r}}\,\mathbf{\Xi}(\boldsymbol{k},\omega)\,d\boldsymbol{k} \tag{92}$$

The dispersion relationship for the wave is derived from Eq. (90) as

$$|\omega - \boldsymbol{v}\cdot\boldsymbol{k}| = ck. \tag{93}$$

This means that the frequency of the electromagnetic wave is being tuned by the moving velocity of the media, e.g. the Doppler effect. This may have an impact on the signals received on earth for a flying object, especially for high-precision signals.

Summary

In this paper, starting from the integral form of the Maxwell's equations, by assuming that the volume and boundary of the dielectric medium are independent of time, the standard differential form of Maxwell's equations was derived. This means that the traditionally known and mostly widely used Maxwell's equations in text book are applicable only to the cases that the medium boundary and volumes are fixed!

For a case in which the medium movement is assumed as a rigid translation in space, we have derived the differential form of expanded Maxwell's equations for this case. With considering the existence of electrostatic charges on the surfaces due to effects such as triboelectrification, the equations are further modified to include various contributions to the displacement current. General strategies for solving the expanded Maxwell's equations are presented using the vector and scalar potentials, which can be further derived using perturbation theory or iteration methods. The expanded Maxwell's equations not only largely expand their applications in various fields, but also serve as the fundamental theory of the nanogenerators including output current and associated electromagnetic radiation. We may speculate that the expanded Maxwell's equations can be applied for calculating the electromagnetic wave radiation in space by a

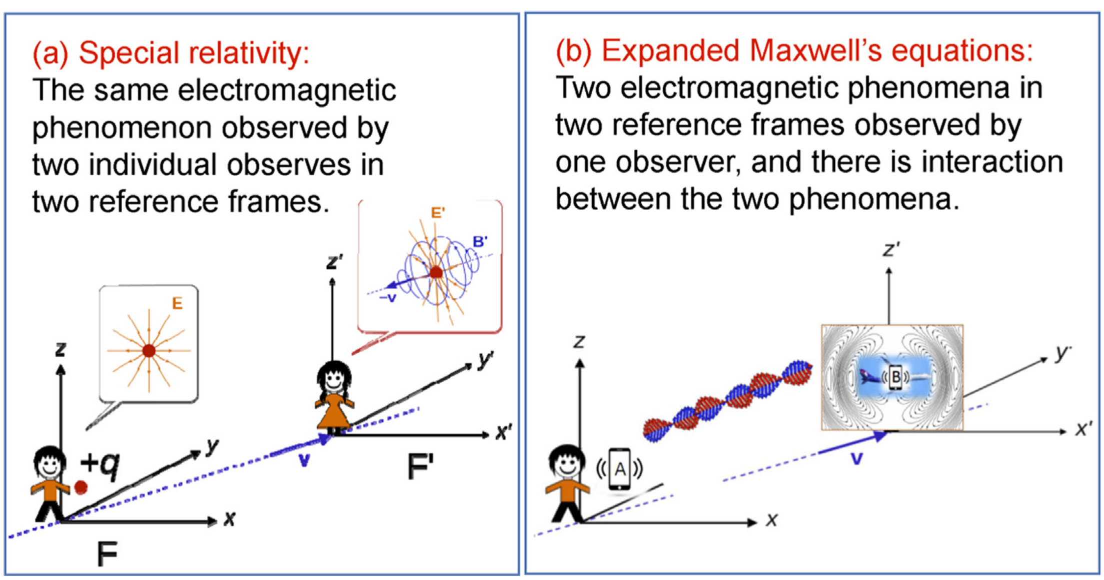

FIGURE 7

Relationship between the theory of special relativity (a) from the expanded Maxwell's equations theory (b). There may have interaction and charges/current exchange between the two electromagnetic events in two separated inertia frames, with one at stationary and the other one is moving.

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flying object/star in space. The correction introduced by the speed term to both the electric field and magnetic field can be more pronounced if the moving speed of the object is high. Such corrections can be predicted using the newly derived equations.

It is important to point out the differences of our expansion of Maxwell's equations presented here from special relativity (Fig. 7). Special relativity is the theory of how different observers, moving at constant velocity with respect to one another, report their experience of the same physical event (Fig. 7a). This description is completely accurate in understanding, except that the special relativity radically altered physicists' new understanding about the unification of space and time. Our expanded Maxwell's equation theory presented here is about the observation of the electromagnetic behavior of the system in a stationary coordination frame when some of the media in the system are moving at a constant speed, and different media could move at different speeds; and there may have interaction and charges/current exchange between the media that are at rest and in moving. In other words, the electromagnetic events in different reference frames are not independent events, but have mutual interaction and energy exchange. Take Fig. 2b as an example, in which medium A remains stationary, medium B is moving, and the observation is performed at the coordination frame of O, and there could be charges or current exchange between medium A and medium B, such as the case for TENG shown in Fig. 1.

Besides TENG, the current theory may be possibly applied to describe the electromagnetic wave generation, transmission, scattering and reflection behavior of moving trains/cars, flight jets, missiles, space shuttles, comet, and even galaxy stars, simply because the electrostatic charges on their surfaces will introduce additional contributions to their electromagnetic behavior if observed on earth. Such studies may have a broad impact on wireless communication and precision signal processing, especially if the phase information of the electromagnetic waves is requied for high resolution imaging.

Data availability statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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